

HOMEWORK 5

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Notation. When we write the metric $\rho(x, a)$, we usually write it as $|x - a|$.

1. # 10.2.3

Proof. “ \Rightarrow ”. Let a be a cluster point for some $E \subset X$: for any $\frac{1}{n} > 0$, $B_{\frac{1}{n}}(a)$ contains infinitely many points. Choosing $x_n \neq a$ such that

$$|x_n - a| < \frac{1}{n}.$$

Thus

$$\lim_{n \rightarrow \infty} x_n = a.$$

“ \Leftarrow ”. If there exists $x_n \in E \subset \{a\}$ such that $\lim_{n \rightarrow \infty} x_n = a$: for any $r > 0$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$,

$$|x_n - a| < r.$$

Therefore $\{x_n\}_{n \geq N} \in B_r(a)$. Therefore a is a cluster point for some $E \subset X$.

□

2. # 10.2.8

Proof. If $f_n \in \mathbb{C}[a, b]$, $f_n \rightarrow f$ uniformly on $[a, b]$. Then $f \in \mathbb{C}[a, b]$. Therefore for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$,

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in [a, b]$. Then

$$\sup_{x \in [a, b]} |f_n(x) - f(x)| \leq \epsilon.$$

Then $\rho(f_n, f) \leq \epsilon$. Therefore $\lim_{n \rightarrow \infty} f_n = f$ in the metric of $\mathbb{C}[a, b]$. The converse direction follows from reversing the reasoning above. □

3. # 10.2.9

Proof. **(a).** $\{x_n\}$ is bounded: $\{x_n\}$ contains a convergent subsequence $\{x_{n_k}\}$ converging to $a \in X$. Since $\{x_{n_k}\} \in E$ and E is closed, then $a \in E$.

(b). If f is bounded on E , for any $n \geq N$, there exists $\{x_n\} \in E$ such that

$$|f(x_n)| > n.$$

By **(a)**, there exists $\{x_{n_k}\} \in E$ such that $x_{n_k} \rightarrow a \in E$,

$$|f(x_{n_k})| > n_k.$$

Sending $k \rightarrow \infty$ to obtain $|f(a)| = \infty$. A contradiction.

(c). From **part (b)**, $m = \inf\{f(x) : x \in E\}$ and $M = \sup\{f(x) : x \in E\}$ exist. By the definition of the infimum and supremum, there exist $\{x_n\} \in E$ and $\{y_n\} \in E$ such that

$$m = \lim_{n \rightarrow \infty} f(x_n), \quad M = \lim_{n \rightarrow \infty} f(y_n).$$

By **part (a)**, there exist $\{x_{n_k}\}$ and $\{y_{n_k}\}$ such that $x_{n_k} \rightarrow x_m$ and $y_{n_k} \rightarrow x_M$. Then because f is continuous,

$$m = f(x_m), \quad M = f(x_M).$$

□

4. # 10.3.5

Proof. The proof is similar to Exercise # 8.3.9 and hence is skipped. □

5. # 10.3.8

Proof. “ \Rightarrow .” V is open in Y : for any $x \in V$, there exists an open ball $B_Y(x, r)$ with $r > 0$ such that $x \in B_Y(x, r) \subset Y$. Write

$$B_Y(x, r) = \{y \in Y : |y - x| < r\} = B_X(x, r) \cap X,$$

where

$$B_X(x, r) = \{y \in X : |y - x| < r\}$$

Therefore

$$V = \cup_{x \in V} B_Y(x, r) = (\cup_{x \in V} B_X(x, r)) \cap X =: U \cap X,$$

where U is open in X .

“ \Leftarrow .” For any $x \in V$, $x \in U$, there exists $B_X(x, r) \subset U$. Then

$$B_X(x, r) \cap Y = B_Y(x, r) \subset V.$$

Therefore V is open in Y .

(b).

“ \Rightarrow ”. Let $E \subset Y$. E is closed in Y . Then E^c is open in Y . By part (b), there exists U open in X ,

$$E^c = U \cap Y.$$

This implies that

$$E = E \cap Y = (U \cap Y)^c \cap Y = U^c \cap Y.$$

Let $A = U^c$. Then A is closed in X .

“ \Leftarrow .” Let $E = A \cap Y$. Then $E \subset Y$. Therefore $E^c = A^c \cup Y^c$. Therefore

$$E^c \cap Y = A^c \cap Y.$$

Therefore $E^c \cap Y$ is open in Y . Since $E \subset Y$, then E is closed in Y . \square

6. # 10.4.1

Proof. **(a).** E is compact since E is bounded and closed. That E is bounded is obvious. To prove that E is closed, let $x_n \in E$ and $x_n \rightarrow a$ as $n \rightarrow \infty$, then $a = 0 \in E$. Otherwise, if $a > 0$, then for large n , $x_n = a$. This implies that

$$a \in E.$$

The option that $a < 0$ is impossible because $x_n \in E$ and $x_n \geq 0$.

(b). The set is bounded and closed. Hence it is compact.

(c). The set is not compact. If so, it should be closed. But taking

$$(x_n, y_n) = \left(\frac{1}{2n\pi}, \sin 2n\pi\right) \rightarrow (0, 0)$$

as $n \rightarrow \infty$. But $(0, 0)$ does not belong to the set.

(d). It is not compact because it is not bounded; if $|xy| \leq 1$, x can be arbitrarily large. \square

7. # 10.4.3

Proof. The proof is skipped. \square

8. # 10.4.5

Proof. \mathbb{R} is a separable metric space. Let V be an open set in \mathbb{R} . Let

$$V = \cup_{j \in \mathbb{N}} B_j,$$

where B_j are open balls in \mathbb{R} such that $B_j = \{x : |x - x_j| < r_j\} = (x_j - r_j, x_j + r_j)$ is an open interval. \square

9. # 10.4.7

Proof. If A and B are compact subsets of X , and X has the Bolzano-Weierstrass property. Then A and B are closed and bounded. If $x \in A, y \in B$,

$$\rho(x, y) \geq \text{dist}(x, B) = \inf\{\rho(x, z) : z \in B\}.$$

Let $f(x) = \text{dist}(x, B)$. Since B is closed, $f(x) > 0$ for $x \in A$. We claim that f is continuous on A .

For $y \in B, x_1, x_2 \in A$,

$$|\rho(x_1, y) - \rho(x_2, y)| \leq \rho(x_1, x_2).$$

Then

$$\rho(x, y) \leq \rho(x_2, y) + \rho(x_1, x_2)$$

which implies

$$f(x_1) \leq \rho(x_2, y) + \rho(x_1, x_2) \text{ for all } y.$$

Then

$$f(x_1) \leq f(x_2) + \rho(x_1, x_2).$$

Similarly

$$f(x_2) \leq f(x_1) + \rho(x_1, x_2).$$

So together we have

$$|f(x_1) - f(x_2)| \leq \rho(x_1, x_2).$$

Therefore f is continuous on A . Since A is compact, by the extreme value theorem,

$$f(x_0) = \inf\{f(x) : x \in A\}$$

exists for some $x_0 \in A$. Therefore

$$\rho(x, y) \geq f(x) \geq f(x_0) > 0$$

for all $x \in A$ and $y \in B$. Then

$$\inf\{\rho(x, y) : x \in A, y \in B\} \geq f(x_0) > 0.$$

This implies that

$$\text{dist}(A, B) \geq f(x_0) > 0.$$

A counterexample if boundedness is dropped. Let $A = \{(x, 1) : x \geq 0\}$ and $B = \{(x, y) : y = \frac{x}{1+x}, x \geq 0\}$. Then both A and B are closed sets. To prove that B is closed: suppose that $(x_n, y_n) \rightarrow (x_0, y_0)$. Then

$$x_n \rightarrow x_0, \text{ and } y_n \rightarrow y_0$$

as n goes to infinity. Since $y_n = \frac{x_n}{1+x_n}$, then as n goes to infinity,

$$y_0 = \frac{x_0}{1+x_0}.$$

□

10. # 10.4.8

Proof. (a). Since $\{H_n\}_{n \geq 1}$ is a nested intervals and $H_i \neq \emptyset$, we prove by contradiction that $\bigcap_{n \geq 1} H_n \neq \emptyset$. Otherwise if $\bigcap_{k \geq 1} H_k = \emptyset$, then

$$X = \bigcup_{k \geq 1} H_k^c.$$

Then $H_1 \subset \bigcup_{k \geq 1} H_k^c$, this implies that

$$H_1 \subset \bigcup_{k \geq 2} H_k^c.$$

Then by compactness of H_1 , there exists $N \in \mathbb{N}$ such that

$$H_1 \subset \bigcup_{j=1}^N H_{k_j}^c = H_{k_N}^c.$$

Thus

$$H_1 \cap H_{k_N} = \emptyset.$$

On the other hand, by the nested property of $\{H_i\}_{i \geq 1}$, we have

$$H_1 \cap H_{k_N} = H_{k_N} \neq \emptyset.$$

A contradiction is obtained.

(b). $(\sqrt{2}, \sqrt{3}) \cap Q$ is bounded. Next we prove that it is closed. Let

$$r_n \in (\sqrt{2}, \sqrt{3}) \cap Q, \text{ and } r_n \rightarrow a \in Q.$$

By the limit comparison theorem,

$$\sqrt{2} \leq a \leq \sqrt{3}.$$

Since $a \in Q$,

$$\sqrt{2} < a < \sqrt{3},$$

which implies that

$$a \in (\sqrt{2}, \sqrt{3}) \cap Q.$$

Therefore $Q \cap (\sqrt{2}, \sqrt{3})$ is bounded and closed. But it is not compact in the metric of Q . Let $\{a_n\}_{n \geq 1} \in (\sqrt{2}, \sqrt{3}) \cap Q$ satisfying $r_1 < r_2 < \dots < r_n < \dots$. Let $r_0 = 0$. Let $R_j = \min\{\frac{r_{j+1}-r_j}{2}, \frac{r_j-r_{j-1}}{2}\}$ for $j \geq 1$. Then firstly

$$(\sqrt{2}, \sqrt{3}) \cap Q \subset \bigcup_{j \geq 1} B(r_j, R_j).$$

Secondly, we prove that the balls are disjoint. Suppose that $j < k$, $B(r_j, R_j) \cap B(r_k, R_k) = \emptyset$. If not, let $x \in B(r_j, R_j) \cap B(r_k, R_k)$. Then $x \in B(r_j, R_j)$ implies that

$$|x| < r_j + R_j \leq r_j + \frac{r_{j+1} - r_j}{2} = \frac{r_j + r_{j+1}}{2};$$

and $x \in B(r_k, R_k)$ implies that

$$|x| > r_k - R_k \geq r_k - \frac{r_k - r_{k-1}}{2} = \frac{r_k + r_{k-1}}{2}.$$

Since $k > j$, then $k - 1 \geq j$. Then

$$\frac{r_k + r_{k-1}}{2} \geq \frac{r_j + r_{j-1}}{2}.$$

Then

$$|x| > |x|.$$

A contradiction. This shows that the balls are disjoint.

If $(\sqrt{2}, \sqrt{3}) \cap Q$ is compact, then there exists $N \in \mathbb{N}$ such that

$$(\sqrt{2}, \sqrt{3}) \cap Q \subset \cup_{k \geq 1} B(r_{j_k}, R_{j_k}).$$

Since the balls are disjoint, there exists $r \in (\sqrt{2}, \sqrt{3}) \cap Q$ such that

$$r \notin \cup_{k=1}^N B(r_j, R_{j_k}).$$

(c). For $k \in N$,

$$H_k = (\sqrt{2} - \frac{1}{k}, \sqrt{2} + \frac{1}{k}) \cap Q.$$

By part (b), H_k is closed and bounded for each k , and $\{H_k\}$ is nested,

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n \subseteq$$

If Cantor's nested property holds for nonempty, nested, closed and bounded sets, then

$$\cap_{k=1}^{\infty} H_k \neq \emptyset.$$

On the one hand,

$$\cap_{k=1}^{\infty} H_k = \{\sqrt{2}\}.$$

On the other hand,

$$\cap_{k=1}^{\infty} H_k \subset Q.$$

A contradiction is obtained. Therefore the claim in (c) holds.

□

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