HOMEWORK 6

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1. # 10.4.5

Proof. If E is a connected set in $\mathbb{R}$, then E is an interval. An interval in $\mathbb{R}$ has the following four forms,

$$[a, b], (a, b), [a, b), (a, b],$$

where a, b are extended real numbers. Then $E^0 = (a, b)$ and so is an interval. Therefore E^0 is connected.

In $\mathbb{R}^2$, this claim fails. Consider Example 10.35. E is connected but E^0 is not connected. To prove that E is connected. Suppose not. There exists two nonempty, and relatively open sets U and V in $\mathbb{R}^2$ such that

$$E = U \cup V, U \cap V = \emptyset.$$

Suppose that $a \in U$ and $b \in V$. We consider

$$F = \{at + b(1-t) : 0 \le t \le 1\}.$$

Since F is a continuous image of $[0, 1]$, E is connected. Consider

$$U_0 = U \cap F, V_0 = V \cap F.$$

Then U_0 and V_0 are not empty because $a \in U_0$ and $b \in V_0$. We also have $U_0 \cap V_0 = \emptyset$, and $F = U_0 \cup V_0$. Finally we claim that U_0 is relatively open in F. Since U is relatively open in E, there exists an open set $A \subset \mathbb{R}^2$ such that

$$U = A \cap E.$$

Therefore

$$U_0 = A \cap E \cap F = A \cap F.$$

Thus U_0 is relatively open in F. Similarly V_0 is relatively open in F. Therefore F is separated by U_0 and V_0. A contradiction. Therefore E is connected.

Note that $E^0 = U \cup V$, where

$$U = \{(x, y) : -1 \le x < 0, -|x| < y < |x|\}, \quad V = \{(x, y) : 0 < x \le 1, -|x| < y < |x|\}.$$

U and V are relatively open, and nonempty in E^0, and $U \cap V = \emptyset$. Therefore E^0 is not connected. □

2. # 10.5.6

Proof. Note that $X \subset \cup_{x\in X} B_x$, and X is compact. Then there exist $x_1, \cdots, x_n \in X$ such that

$$X = \cup_{k=1}^{n} B_k(x_k).$$

Suppose that $y_i = f(x_i), 1 \leq i \leq n$. If all the y_i are the same, then f is constant. Otherwise after relabeling, there exists $1 \leq N \leq n$ such that

$$\{y_1, \cdots, y_n\} = \{\tilde{y}_1, \cdots, \tilde{y}_N\},$$

and those $\tilde{y}_i, 1 \leq i \leq N$ are disjoint. Let

$$E_i = f^{-1}(\tilde{y}_i) = \{x \in X : f(x) = \tilde{y}_i\}.$$

It is not hard to prove that E_i is open. Obviously

$$X = E_1 \cup \{E_2 \cup \cdots \cup E_N\} = U \cup V.$$

We know that U and V is open in X, $U \cap V = \emptyset$, and U, V are not empty. Therefore X is not connected. This is a contradiction. So the proof that f is constant is complete. □

3. # 10.5.7

Proof. **"⇒".** Each E_α is relatively open in H, there exists U_α open in X such that

$$E_\alpha = U_\alpha \cap H.$$

So

$$H \subset \cup_\alpha E_\alpha = (\cup_{\alpha\in A} U_\alpha) \cap H \subset \cup_{\alpha\in A} U_\alpha.$$

Since H is compact in X, there exists N such that for $1 \leq i \leq N$,

$$H \subset \cup_{1\leq i\leq N} U_i.$$

Intersecting with H,

$$H \subset \cup_{1\leq i\leq N}(U_i \cup H) = \cup_{1\leq i\leq N} E_i.$$

So $\{E_\alpha\}$ has a finite sub-cover.

"⇐". For any open covering V_α of H, V_α is open in X and hence is relatively open in H. By the hypothesis, there exists a finite cover of H. Hence H is compact. □

4. # 10.6.1

Proof. The computation is skipped. □

5. # 10.6.3

Proof. This follows from Theorem 10.58 and the following fact, for any closed set C in Y, $C = O^c$ where O is an open set in Y. Then

$$f^{-1}(C) = \left(f^{-1}(O)\right)^c.$$

□

6. # 10.6.5

Proof. Since E is connected in $\mathbb{R}$ and f is continuous, then $f(E)$ is connected in $\mathbb{R}$. By theorem 10.56, suppose that $f(E) = (c, d)$. Suppose that $f(a) < f(b)$ and $a \le y \le f(b)$. Then $y \in [f(a), f(b)] \subset f(E)$. Since $y \in f(E)$, there exists $x \in E$ such that

$$y = f(x).$$

□

7. # 10.6.6

Proof. **(a).** This is proven by composing two continuous functions,

$$f : H \to Y, \quad g = \|\cdot\|_Y : Y \to \mathbb{R}$$

and using Theorem 10.63. It is left as an exercise that the composition of two continuous functions is continuous on the general metric spaces. Another proof follows similarly from Theorem 10.6.3.

(b). The proof is skipped.

(c). Here we only prove the backward implication. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$k, j \ge N, \Rightarrow \|f_k - f_j\|_H < \epsilon.$$

i.e.,

$$\|f_k(x) - f_j(x)\|_Y \le \epsilon$$

for all $x \in X$. Then for each x, $\{f_k(x)\}$ is a Cauchy sequence in an Euclidean space and hence it is convergent to some point in Y, denoted by $f(x)$. Sending j to ∞, then

$$\|f_k(x) - f(x)\|_Y \le \epsilon$$

for all $x \in X$. This implies that $\{f_k\}_k$ converges uniformly to f on X. □

8. # 10.6.9

Proof. We prove it by contradiction. Suppose that X has at most countably many points. Then $f(X)$ has at most countably many points in $\mathbb{R}$ by Lemma 1.40. Since f is not constant, $f(X)$ contains at least two points. On the other hand, since f is continuous on X and X is connected, then $f(X)$ is connected in $\mathbb{R}$. By Theorem 10.56, $f(X)$ is an interval. Since it contains at least two points, it is non-degenerate. Since non-degenerate intervals in $\mathbb{R}$ only take the following four forms,

$$[a,b],(a,b),[a,b),(a,b]$$

with $a < b$, where a,b are extended real numbers. None of these intervals is countable. A contradiction is obtained. Therefore X contains uncountably many points. □

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