HOMEWORK 7

SHUANGLIN SHAO

1. # 11.1.2

Proof. **a).** For $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{4x^3}{x^2+y^2} + (x^4+y^4)\frac{-2x}{x^2+y^2}$$
$$= \frac{2x^5+4x^3y^2-2xy^4}{(x^2+y^2)^2},$$

and

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0.$$

For $(x,y) \neq (0,0)$,

$$|f_x(x,y)| \leq \frac{4x^3y^2}{(x^2+y^2)^2} + \frac{2x(x^4-y^4)}{(x^2-y^2)^2} \leq 4|x| + 2|x| \leq 6|x|.$$

Therefore $\lim_{(x,y)\to(0,0)} f_x(x,y) = 0 = f_x(0,0)$.

b). For $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{4x^3+8xy^2}{3\sqrt[4]{(x^2+y^2)^3}},$$

and

$$f_x(0,0) = 0.$$

For $(x,y) \neq (0,0)$,

$$|f_x(x,y)| \leq \frac{8|x|(x^2+y^2)}{3(x^2+y^2)^{4/3}} \leq \frac{8|x|(x^2+y^2)}{3(x^2+y^2)^{1/3}(x^2+y^2)} \leq \frac{8}{3}|x|^{1/3}$$

since $x^2+y^2 \geq x^2$. Therefore $\lim_{(x,y)\to(0,0)} f_x(x,y) = 0 = f_x(0,0)$. □

2. # 11.1.4

Proof. The function g is integrable and hence it is bounded: there exists $M > 0$ such that

$$|g(x)| \leq M.$$

We need to prove that for $y_0 \in [c, d]$,

$$\lim_{y \to y_0} F(y) = F(y_0).$$

The function F is continuous on $H = [a, b] \times [c, d]$ and so it is uniformly continuous on H. For any $\epsilon > 0$, there exists $\delta > 0$ such that for $|(x_1, y_1) - (x_2, y_2)| < \delta$ and $(x_i, y_i) \in H$ for $i = 1, 2$,

$$|F(x_1, y_1) - F(x_2, y_2)| < \epsilon / M(|b - a| + 1).$$

For $|y - y_0| < \delta$,

$$\begin{aligned} \int_a^b g(x) f(x, y) dx - \int_a^b g(x) f(x, y_0) dx \; &\leq \int_a^b |g(x)| \, |f(x, y) - f(x, y_0)| \, dx \\ &< \frac{M|b - a|}{M|b - a| + 1} \epsilon \\ &< \epsilon, \end{aligned}$$

which implies that f is uniformly continuous on H. □

3. # 11.1.5

Proof. For this exercise, we apply Theorem 11.4 and Theorem 11.5.

(a). The function $e^{x^3 y^2 + x}$ is continuous on $H = [0, 1] \times [0, 1]$. Therefore

$$\lim_{y \to 0} \int_0^1 e^{x^3 y^2 + x} dx = \int_0^1 e^x dx = e^x |_0^1 = e - 1.$$

(b). The function $\sin(e^x y - y^3 + \pi - e^x)$ is continuous on $H = [0, 1] \times [0, 1]$ and the derivative $\frac{d}{dy} \left(\sin(e^x y - y^3 + \pi - e^x) \right) = \cos(e^x y - y^3 + \pi - e^x)(e^x - 3y^2)$. Therefore at $y = 1$,

$$\begin{aligned} &\frac{d}{dy} \int_0^1 \sin(e^x y - y^3 + \pi - e^x) dx \\ &= \int_0^1 \frac{d}{dy} \sin(e^x y - y^3 + \pi - e^x) dx \\ &= \cos(\pi - 1)(e - 4). \end{aligned}$$

(c). The computation is similar as in (b). □

4. # 11.2.2

Proof. In $\mathbb{R}^m$, for $x, a \in \mathbb{R}^m$, for $f : \mathbb{R} \to \mathbb{R}^m$,

$$\lim_{x \to a} \|f(x)\| = \| \lim_{x \to a} f(x) \|.$$

This is because taking norm in $\mathbb{R}^m$ is taking squares of a vector in $\mathbb{R}^{\gtrdot}$, and then does inverting-squares. Since $f(a) = g(a) = 0$,

$$\lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{\|x - a\|}}{\dfrac{g(x) - g(a)}{\|x - a\|}} = \lim_{x \to a} \frac{\dfrac{\epsilon(x-a)}{\|x-a\|} + Df(a)\dfrac{x-a}{\|x-a\|}}{\dfrac{\epsilon(x-a)}{\|x-a\|} + Dg(a)\dfrac{x-a}{\|x-a\|}}.$$

Since f and g are from $\mathbb{R}$ to $\mathbb{R}^m$, then

$$Df(a) = (u_1, u_2, \cdots, u_m), \quad Dg(a) = (v_1, v_2, \cdots, v_m).$$

Also write $\frac{\epsilon(x-a)}{\|x-a\|}$ as $(w_1, \cdots, w_m)$. Then

$$\frac{\epsilon(x-a)}{\|x-a\|} + Df(a)\frac{x-a}{\|x-a\|} = \sqrt{\sum_{i=1}^{m} |w_i|^2 + \sum_{i=1}^{m} u_i \frac{x-a}{\|x-a\|}^{2}}.$$

Since $\lim_{x \to a} \dfrac{\epsilon(x-a)}{\|x-a\|} = 0$,

$$\lim_{x \to a} \sum_{i=1}^{m} |w_i|^2 = 0.$$

Then by the limit theorem, as $x \to a$, the right hand side equals

$$\sqrt{\lim_{x \to a} \sum_{i=1}^{m} u_i \frac{x-a}{\|x-a\|}^{2}} = \sqrt{\lim_{x \to a} \sum_{i=1}^{m} |u_i| \times \frac{x-a}{\|x-a\|}^{2}} = \|Df(a)\|,$$

since $|x - a| = \|x - a\|$ on $\mathbb{R}$. This establish the claim

$$\lim_{x \to a} \frac{\dfrac{f(x) - f(a)}{\|x - a\|}}{\dfrac{g(x) - g(a)}{\|x - a\|}} = \frac{\|Df(a)\|}{\|Dg(a)\|}.$$

$\square$

5. # 11.2.3

Proof. The derivative does not exist. Suppose it exists. Then the total derivative of f at $(0,0)$ is

$$Df(0,0) = (\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)) = (0,0).$$

Therefore for $h \in \mathbb{R}^2$ satisfying that $h_1 > 0$ and $h_2 = kh_1$ with $h_1 > 0$,

$$\frac{f(h) - f(0) - Df(0,0)\cdot h}{\sqrt{h_1^2 + h_2^2}} = \frac{\sqrt{h_1 h_2}}{\sqrt{h_1^2 + h_2^2}} = \frac{\sqrt{k}}{\sqrt{1+k^2}}.$$

This depends on the values of k. So the derivative of f does not exist. □

6. # 11.2.4

Proof. To compute the partial derivative,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{\frac{h^2}{\sin|h|} - 0}{h} = \lim_{h\to 0} \frac{h}{\sin|h|}.$$

Since $\lim_{h\to 0} \frac{\sin h}{h} = 1$. For $h > 0$, the limiting value is 1; for $h < 0$, the limiting value is -1. So the partial derivative of f does not exist. Therefore the total derivative of f does not exist at $(0,0)$. □

7. # 11.2.5

Proof. We compute that, for $(x,y) \neq (0,0)$,

$$\frac{\partial f}{\partial x}(x,y) = \frac{4x^3(x^2+y^2) - 2\alpha x(x^4+y^4)}{(x^2+y^2)^{\alpha+1}},$$

and

$$\frac{\partial f}{\partial y}(x,y) = \frac{4y^3(x^2+y^2) - 2\alpha y(x^4+y^4)}{(x^2+y^2)^{\alpha+1}}.$$

These two partial derivatives are continuous at all $(x,y) \in \mathbb{R}^2 \setminus (0,0)$.

At $(0,0)$,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{h^{4-2\alpha}}{h} = 0,$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{k^{4-2\alpha} - 0}{k} = 0.$$

We prove that the partial derivatives are continuous at $(0,0)$. Since $x^4+y^4 \geq \frac{(x^2+y^2)^2}{2}$,

$$\frac{\partial f}{\partial x}(x,y) \leq \frac{4|x|^3}{(x^2+y^2)^2} + 2\alpha|x|(x^2+y^2)^{1-\alpha} \leq C|x|^{3-2\alpha}$$

for all $1 \leq \alpha < \frac{3}{2}$.

For all $0 \leq \alpha < 1$,

$$\frac{\partial f}{\partial x}(x,y) \leq C|x|^{3-2\alpha} + 2\alpha|x|(x^2+y^2)^{1-\alpha},$$

which goes to zero as $(x,y) \to (0,0)$.

For all $\alpha < 0$,

$$\frac{\partial f}{\partial x}(x,y) \leq 4|x|(x^2+y^2)^{-\alpha} + 2\alpha|x|(x^2+y^2)^{1-\alpha},$$

which goes to zero as $(x,y) \to (0,0)$. Then the partial derivatives are continuous at $(0,0)$.

Therefore by Theorem 11.15, f is differentiable on $\mathbb{R}^2$. □

8. # 11.2.8

Proof. For $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, let

$$f(x) = Tx.$$

We write $x \in \mathbb{R}^n$ as a column vector. Then we compute

$$\epsilon(h) = f(a+h) - f(a) - Th = T(a+h) - Ta - Th = 0.$$

Therefore $\frac{\epsilon(h)}{\|h\|} \to 0$ as $h \to 0$. □

9. # 11.2.11

Proof. **(a).** For a, b, h, define

$$F(h) := f(a+h, b+h) - f(a+h, b), \text{ for } |h| < \frac{r}{\sqrt{2}}.$$

Then by the mean value theorem, there exists $t = t(a,b,h)$,

$$F(h) - F(0) = F'(t)h.$$

This implies that

$$\frac{\Delta(h)}{h} = f_y(a+h, b+th) - f_y(a, b+th).$$

Therefore by adding terms and subtracting terms, the claim is established since

$\nabla f_y(a,b) = (f_{yx}(a,b), f_{yy}(a,b)), \nabla f_y(a,b)\cdot(h,th) - \nabla f_y(a,b)\cdot(0,th) = hf_{yx}(a,b).$

(b). Let

$\epsilon_1(h) = f_y(a+h, b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(h,th), \text{ for } 0 < |h| < r, \epsilon_1(0) = 0.$

For $h > 0$, we write

$$\begin{aligned}\frac{\epsilon_1(h)}{h} &= \frac{f_y(a+h,b+th) - f_y(a,b) - \nabla f_y(a,b)(h,th)}{\sqrt{1+t^2}h} \times \frac{\sqrt{1+t^2}h}{h} \\ &\leq \frac{f_y(a+h,b+th) - f_y(a,b) - \nabla f_y(a,b)(h,th)}{\sqrt{1+t^2}h}\end{aligned}$$

Let $h \to 0$, since f_y is differentiable at (a,b), it equals 0.Then

$$\lim_{h\to 0^+} \frac{\epsilon_1(h)}{h} = 0.$$

Similarly define

$\epsilon_2(h) = f_y(a, b+th) - f_y(a,b) - \nabla f_y(a,b)\cdot(0,th), \text{ for } 0 < |h| < r, \epsilon_1(0) = 0.$

For $h > 0$, we write

$$\begin{aligned}\frac{\epsilon_2(h)}{h} &= \frac{f_y(a,b+th) - f_y(a,b) - \nabla f_y(a,b)(0,th)}{th} \times \frac{th}{h} \\ &\leq \frac{f_y(a,b+th) - f_y(a,b) - \nabla f_y(a,b)(0,th)}{th}\end{aligned}$$

Let $h \to 0$, since f_y is differentiable at (a,b), so it equals 0. Therefore

$$\lim_{h\to 0^+} \frac{\epsilon_2(h)}{h} = 0.$$

These two limits imply

$$\lim_{h\to 0^+} \frac{\Delta(h)}{h^2} = f_{yx}(a,b).$$

Similarly

$$\lim_{h\to 0^-} \frac{\Delta(h)}{h^2} = f_{yx}(a,b).$$

So

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a,b).$$

(c). Similarly the argument in (a) and (b) implies that

$$\lim_{h\to 0} \frac{\Delta(h)}{h^2} = f_{xy}(a,b).$$

Then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

□

Department of Mathematics, KU, Lawrence, KS 66045

E-mail address: slshao@math.ku.edu