HOMEWORK 8

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1. # 11.3.1

Proof. **(a).** For $f(x,y) = x - y$, and $g(x,y) = x^2 + y^2$,

$$Df(x,y) = (1,-1), \text{ and } Dg(x,y) = (2x, 2y).$$

Then

$$D(f+g)(x,y) = Df(x,y) + Dg(x,y) = (2x+1, 2y-1);$$

and

$$D(f \cdot g)(x,y) = gDf(x,y) + fDg(x,y) = (3x^2 + y^2 - 2xy, -x^2 + 2xy - 3y^2).$$

(b).

$$Df(x,y) = (y,x), \text{ and } Dg(x,y) = (\sin x - x\cos x, -\sin y).$$

Then at (x,y),

$$D(f+g)(x,y) = (y + \sin x - x\cos x, x - \sin y);$$

and

$$\begin{aligned} D(f \cdot g)(x,y) &= gDf(x,y) + fDg(x,y) \\ &= (2xy\sin x - y\cos y - x^2 y\cos x, x^2\sin x - xy\sin y - x\cos y). \end{aligned}$$

□

2. # 11.3.2

Proof. **(a).** The gradient of f at $(1,-1)$ is

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (2, -2).$$

Then the tangent plane is

$$2x - 2y - z = 2.$$

(b). The gradient of f at $(1,1)$ is

$$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (2, -2).$$

Then the tangent plane is

$$2x - 2y - z = 0.$$

□

3. # 11.4.1

Proof. For $w = F(f, g, h)$, then

$$w_p = \frac{\partial F}{\partial x}\frac{\partial f}{\partial p} + \frac{\partial F}{\partial y}\frac{\partial g}{\partial p} + \frac{\partial F}{\partial z}\frac{\partial h}{\partial p};$$

and

$$w_q = \frac{\partial F}{\partial x}\frac{\partial f}{\partial q} + \frac{\partial F}{\partial y}\frac{\partial g}{\partial q} + \frac{\partial F}{\partial z}\frac{\partial h}{\partial q};$$

and

$$\begin{aligned} w_{pp} = & \left(\frac{\partial^2 F}{\partial x^2}(\frac{\partial f}{\partial p})^2 + \frac{\partial^2 F}{\partial x \partial y}\frac{\partial f}{\partial p}\frac{\partial g}{\partial p} + \frac{\partial^2 F}{\partial x \partial z}\frac{\partial f}{\partial p}\frac{\partial h}{\partial p}\right) + \frac{\partial F}{\partial x}\frac{\partial^2 f}{\partial p^2} \\ & + \left(\frac{\partial^2 F}{\partial x \partial y}\frac{\partial f}{\partial p}\frac{\partial g}{\partial p} + \frac{\partial^2 F}{\partial y^2}(\frac{\partial g}{\partial p})^2 + \frac{\partial^2 F}{\partial y \partial z}\frac{\partial g}{\partial p}\frac{\partial h}{\partial p}\right) + \frac{\partial F}{\partial z}\frac{\partial^2 h}{\partial p^2} \\ & + \left(\frac{\partial^2 F}{\partial z \partial x}\frac{\partial f}{\partial p}\frac{\partial h}{\partial p} + \frac{\partial^2 F}{\partial z \partial y}\frac{\partial g}{\partial p}\frac{\partial h}{\partial p} + \frac{\partial^2 F}{\partial z^2}(\frac{\partial h}{\partial p})^2\right) + \frac{\partial F}{\partial z}\frac{\partial^2 h}{\partial p^2}. \end{aligned}$$

□

4. # 11.4.4

Proof. For $u = f(xy)$,

$$\frac{\partial u}{\partial x} = yf' \text{ and } \frac{\partial u}{\partial y} = xf'.$$

Then

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial x} = 0.$$

For $v(x, y) = f(x - y) + g(x + y)$,

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = (f'' + g'') - (f'' + g'') = 0.$$

□

5. # 11.4.6

Proof. For $u(r,\theta) = f(r\cos\theta, r\sin\theta)$, then

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta,$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial^2 f}{\partial x^2}(\cos\theta)^2 + \frac{\partial^2 f}{\partial y^2}(\sin\theta)^2 + 2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta,$$

$$\frac{\partial^2 u}{\partial \theta^2} = r^2\left(\frac{\partial^2 f}{\partial x^2}(\sin\theta)^2 + \frac{\partial^2 f}{\partial y^2}(\cos\theta)^2 - 2\frac{\partial^2 f}{\partial x\partial y}\sin\theta\cos\theta\right) - r\frac{\partial f}{\partial x}\cos\theta - r\frac{\partial f}{\partial y}\sin\theta.$$

Then since $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$,

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = 0.$$

□

6. # 11.4.10

Proof. For all $t \in I$,

$$r^2 = \|f(t)\|^2.$$

Differentiating f in t, then

$$2\sum_{k=1}^{m} f_k(t)f_k'(t) = 0.$$

Then $f(t)$ is orthogonal to $f'(t)$. □

7. # 11.4.11

Proof. **(a).** The directional derivative $D_u f$ is defined,

$$D_u f(a) := \lim_{t\to 0}\frac{f(a+tu) - f(a)}{t}.$$

Write $g(t) = a + tu$. Then $h(t) = f(g(t)) = f(a+tu)$ and $h(0) = f(a)$. h is differentiable at $t = 0$. Therefore

$$\lim_{t\to 0}\frac{f(a+tu) - f(a)}{t} = \lim_{t\to 0}\frac{h(t) - h(0)}{t} = h'(0).$$

Then

$$h'(0) = Df(g(0))g'(0) = \nabla f(a)\cdot u.$$

(b). By (a),

$$D_u f(a) = \nabla f(a)\cdot u = \|\nabla f(a)\|\cos\theta,$$

where θ is the angle between $\nabla f(a)$ and u.

(c). By (b), when $\theta = 0$,

$$D_u f(a) = \|\nabla f(a)\|.$$

It is known that when $\theta \in [0, \pi/2]$,

$$0 \leq D_u f(a) \leq \|\nabla f(a)\|.$$

Therefore the maximum of $D_u f(a)$ is $\|\nabla f(a)\|$, which occurs when u is parallel to $\nabla f(a)$. □

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