HOMEWORK 9

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1. # 11.5.1

Proof. **(a).** By using Theorem 11.37, $f(a) = 1$,
$f(x,y) = 1 + (-1,1)\cdot(x+1,y-1) + \left((x+1)^2 + (x+1)(y-1) + (y-1)^2\right).$

(c). We compute the derivatives,
$\frac{\partial f}{\partial x} = ye^{xy}, \frac{\partial f}{\partial y} = xe^{xy},$
$\frac{\partial^2 f}{\partial x^2} = y^2e^{xy}, \frac{\partial^2 f}{\partial x \partial y} = xye^{xy} + e^{xy}, \frac{\partial^2 f}{\partial y^2} = x^2e^{xy},$
$\frac{\partial^3 f}{\partial x^3} = y^3e^{xy}, \frac{\partial^3 f}{\partial x^2 \partial y} = xy^2e^{xy} + 2ye^{xy}, \frac{\partial^3 f}{\partial x \partial y^2} = x^2ye^{xy} + 2xe^{xy}, \frac{\partial^3 f}{\partial y^3} = x^3e^{xy},$
$\frac{\partial^4 f}{\partial x^4} = y^4e^{xy}, \frac{\partial^4 f}{\partial x^3 \partial y} = xy^3e^{xy} + 3y^2e^{xy}, \frac{\partial^4 f}{\partial x^2 \partial y^2} = x^2y^2e^{xy} + 2xye^{xy} + 2e^{xy},$
$\frac{\partial^4 f}{\partial x \partial y^3} = x^3ye^{xy} + 3x^2e^{xy}, \frac{\partial^4 f}{\partial y^4} = x^4e^{xy}.$
Therefore by Theorem 11.37, there exists $c = (c_1, c_2)$ such that

$$f(x,y) = 1 + xy + \frac{1}{4!}\left(e^{c_1c_2}c_2^4x^4 + 4(e^{c_1c_2}c_1c_2^3 + 3e^{c_1c_2}c_2^2)x^3y + 6(e^{c_1c_2}c_1^2c_2^2 \right.$$
$$\left. + 4e^{c_1c_2}c_1c_2 + 2e^{c_1c_2})x^2y^2 + 4(e^{c_1c_2}c_1^3c_2 + 3e^{c_1c_2}c_1^2)xy^3 + e^{c_1c_2}c_1^4y^4\right).$$

□

2. # 11.5.2

Proof. This follows directly from Theorem 11.37. For $a = (x_0, y_0)$,

$$\sum_{i_1=1}^{2} \cdots \sum_{i_k=1}^{2} \frac{\partial^k f(a)}{\partial x_{i_1} \cdots \partial x_{i_k}} h_{i_1} \cdots h_{i_k} = \sum_{i+j=k} \binom{k}{i} \frac{\partial^k f(a)}{\partial x^i \partial y^j}(x-x_0)^i(y-y_0)^j.$$

□

3. # 11.5.5

Proof. We prove it by mathematics induction. The case where $k = 1$ follows from the fundamental theorem of Calculus. Suppose it is true for $k \in \mathbb{N}$. Then write

$$(1-t)^{p-1} = -\frac{1}{p}\left((1-t)^p\right)'.$$

Then by the integration by parts,

$$\frac{1}{(p-1)!}\int_0^1 (1-t)^{p-1} D^{(p)} f(a+th;h)dt = -\frac{1}{p!}\int_0^1 (1-t)^p D^{(p)} f(a+th;h)dt$$
$$= \frac{1}{(p-1)!} D^p f(a;h) + \frac{1}{p!}\int_0^1 (1-t)^p D^{p+1} f(a+th;h)dt.$$

Thus the formula for $k+1$ follows. □

4. # 11.5.6

Proof. **(a).**

$$g'(t) = \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b).$$

(b). By the mean value theorem for one-dimensional function, there exists $\theta \in (0,1)$ such that

$$g(1) - g(0) = g'(\theta),$$

i.e.,

$$f(x,y) - f(a,b) = (x-a)f_x(c,y) + (y-b)f_y(a,d),$$

where

$$c = \theta x + (1-\theta)a, d = \theta y + (1-\theta)b.$$

□

5. # 11.5.8

Proof. This follows from Theorem 11.37.

$$\nabla f(a) = 0.$$

Since $f \in \mathbb{C}^2$, for $1 \le i, j \le n$, it implies that $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is continuous on V and hence is bounded on the compact set H. Since there is n^2's $\frac{\partial^2 f}{\partial x_i \partial x_j}$, and

$$|(x_i - a_i)(x_j - a_j)| \le \frac{(x_i - a_i)^2 + (x_j - a_j)^2}{2} \le \|x - a\|,$$

we see that

$$|f(x) - f(a)| \le M\|x - a\|$$

for some $M > 0$. ☐

6. # 11.5.11

Proof. **(a)**. Without loss of generality, suppose that $a < b$ and $c > 0$. Since $u \in \mathbb{C}^2(V)$, u is continuous on V and therefore u is uniformly continuous on H. For any $\epsilon > 0$, there exists $\theta > 0$ satisfying $2\delta < \min\{c, b-a\}$ such that for $\|(x,y) - (x_0, y_0)\| < \delta$, we have

$$|f(x,y) - f(x_0, y_0)| < \epsilon.$$

Let $K = [a + \frac{\delta}{2}, b - \frac{\delta}{2}] \times [\frac{\delta}{2}, c - \frac{\delta}{2}]$. Therefore K is a compact subset of H and $K \subset H^0$. For any $(x,y) \in H \setminus K$, for any $(x_0, y_0) \in \partial H$,

$$f(x,y) \geq f(x_0, y_0) - \epsilon \geq \epsilon.$$

(b). Here $\epsilon = \frac{\epsilon}{2} - rt_1 > 0$. Then by part (a), there exists a compact set $K \subset H^0$ such that

$$w \geq -\epsilon.$$

Since u is continuous on H and H is compact, then there exists $(x_2, t_2) \in H$ such that $u(x_2, t_2)$ is the minimum point. If $(x_2, t_2) \in H \setminus K$, then

$$-l \geq u(x_2, t_2) + rt \geq -\frac{l}{2} + 2rt_1.$$

Hence $2rt_1 \leq -\frac{l}{2} < 0$. A contradiction. So $(x_2, t_2) \in K$.

(c). We prove it by contradiction. Suppose that u attains the minimum at the interior point, $(x_1, t_1) \in H^0$ and $u(x_1, t_1) = -l < 0$ for some $l > 0$. By part (b), the function w attains the minimum at some $(x_2, t_2) \in K \subset H^0$. Therefore

$$w_t(x_2, t_2) = 0, \text{ and } w_{xx}(x_2, t_2) \geq 0.$$

Since u satisfies $u_t = u_{xx}$, then

$$0 = w_t = u_t + r = u_{xx} + r \geq r > 0.$$

We obtain a contradiction. Thus $u(x,t) \geq 0$ on H. ☐

7. # 11.5.12

Proof. **(a).** We prove it by contradiction. If E is separated by two relatively open, nonempty subsets of U and V such that

$$E = U \cup V, \text{ and } U \cap V = \emptyset.$$

Since U and V are not empty and E is a convex set, there exist $x_0 \in U$ and $y_0 \in V$ such that

$$L(x_0, y_0) \subset E.$$

The line segment $L(x_0, y_0) = \{tx_0 + (1-t)y_0 : 0 \le t \le 1\}$ is the continuous image of $[0, 1]$. Hence it is connected.

However $L = (L \cap U) \cup (L \cap V)$ and $(L \cap U) \cap (L \cap V) = \emptyset$. We claim that $L \cap U$ and $L \cap V$ are relatively open sets in L. Since U is relatively open in $\mathbb{R}^n$, there exists an open set $A \subset \mathbb{R}^n$ such that

$$U \cap L = E \cap A \cap L = L \cap A.$$

Then $U \cap L$ is relatively open in L. This proves that L is not connected. A contradiction. The proof that a convex set is connected in $\mathbb{R}^n$ is complete.

(b). We take the following example.

$$E := \{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4.\}$$

Then E is connected but not convex. The proof that E is connected is similar to part (a). The difference that we cannot connect two points $x_0, y_0 \in E$ by a line segment $L(x_0, y_0)$. However, we can choose a polygon line to connect them, and regard them as a continuous image of $[0, 1]$.

But it is clear that E is not convex because $(1, 0), (-1, 0) \in E$ but $(0, 0) \notin E$.

□

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