MATH 810 FINAL

Choose any three to turn in. You may consult with any reference or search online, and discuss with your fellow students. But you have to write it in your words.

Problem 1. (Whitney's decomposition.) Let $F \subset \mathbb{R}^n$ be a given closed set. Then there exists a collection of cubes $\mathcal{F}$, $\mathcal{F} = \{Q_1, Q_2, \cdots, Q_k, \cdots\}$ (closed cube in $\mathbb{R}^n$, with sides parallet to axesams) so that

- $\cup_k Q_k = \Omega = F^c$, where F^c denotes the complement of F.
- The Q_k are mutually disjoint, i.e., their interiors are disjoint.
- $c_1 \,\mathrm{diameter}(Q_k) \leq \mathrm{distance}(Q_k, F) \leq c_2 \,\mathrm{diameter}(Q_k)$. The constants c_1, c_2 are independent of F. In fact, we may take $c_1 = 1, c_2 = 4$.

Problem 2.(Hanner's inequalities.) Let $f, g \in L^p(\mathbb{R}^n)$. If $1 \leq p \leq 2$, we have

$$\begin{aligned} &\|f+g\|_p^p + \|f-g\|_p^p \geq (\|f\|_p + \|g\|_p)^p + |\|f\|_p - \|g\|_p|^p, \\ &(\|f+g\|_p + \|f-g\|_p)^p + |\|f+g\|_p - \|f-g\|_p|^p \leq 2^p(\|f\|_p^p + \|g\|_p^p). \end{aligned} \tag{1}$$

If $2 \leq p < \infty$, the inequalities are reversed.

Problem 3.(Differentiability of norms.) Suppose $f, g \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$. The function defined on $\mathbb{R}$ by

$$N(t) = \int_{\mathbb{R}^n} |f(x) + tg(x)|^p dx$$

is differentiable and its derivative at $t = 0$ is given by

$$\frac{d}{dt} N|_{t=0} = \frac{p}{2} \int_{\mathbb{R}^n} |f(x)|^{p-2} \big(\bar{f}(x) g(x) + f(x) \bar{g}(x)\big) dx.$$

Problem 4.(Missing terms in Fatou's lemma.) Let f^j be a sequence of complex-valued functions on $\mathbb{R}^n$ that converges pointwise a.e. to a function f. Suppose also that for $0 < p < \infty$,

$$\int_{\mathbb{R}^n} |f^j(x)|^p dx < C,$$

for all j and for some constant $C > 0$. Then

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} \big| |f^j(x)|^p - |f^j(x) - f(x)|^p - |f(x)|^p \big| dx = 0.$$

Problem 5. The following are equivalent.

- The function f is continuous on (a, b) and satisfies the midpoint convexity condition
$$f(\frac{1}{2}x + \frac{1}{2}y) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y), \forall x, y \in (a, b).$$
- The function f is convex on (a, b), i.e.,
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \forall x, y \in (a, b), \lambda \in [0, 1].$$

Problem 6. Let E_j be collection of subsets of a fixed compact set in $\mathbb{R}^n$ with $\sum_j |E_j| = \infty$. Then there exists a sequence of translates $F_j = E_j + x_j$ so that

$$\limsup F_j = \cap_{k=1}^{\infty} \big(\cup_{j=k}^{\infty} F_j \big) = \mathbb{R}^n.$$

Problem 7. (Loomis-Whitney's inequality.) Suppose that we have for $j = 1, 2, \cdots, n$,

$$f_j : \mathbb{R}^{n-1} \to \mathbb{R}^+$$

a positive measurable function and

$$\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$$

the projection by forgetting the $j-$th coordinate, then

$$\int_{\mathbb{R}^n} \Pi_{j=1}^n f_j \circ \pi_j dx \leq \Pi_{j=1}^n \|f_j\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$